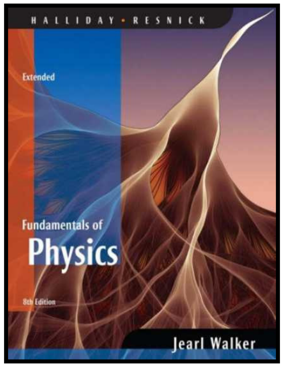
Workshop Physics 1017 - 311

University Physics I



Week 3 : Day 2

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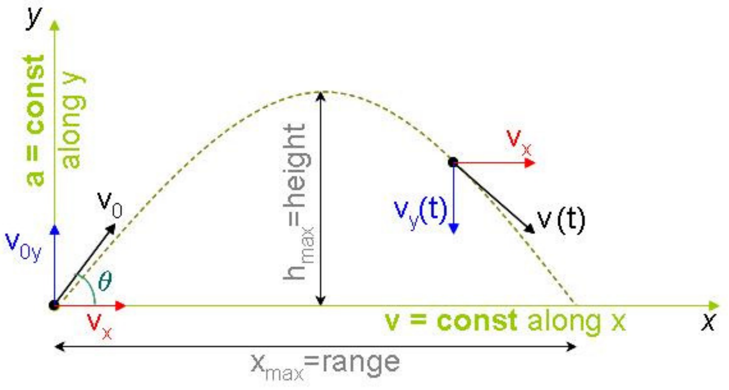
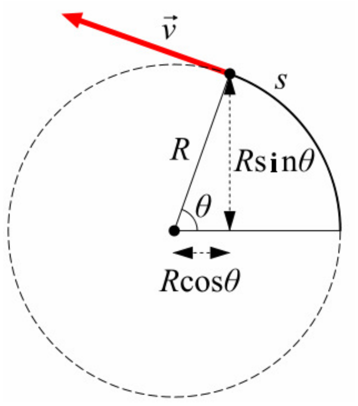
Multi-Dimensional Motion

- **We will examine the generalization of the 1D Kinematics equations for 2D and 3D cases**
  - General Description
    - *Kinematics in 3D*
  - Two Dimensional Motion – Special Cases
    - *Projectile Motion* *Circular Motion*

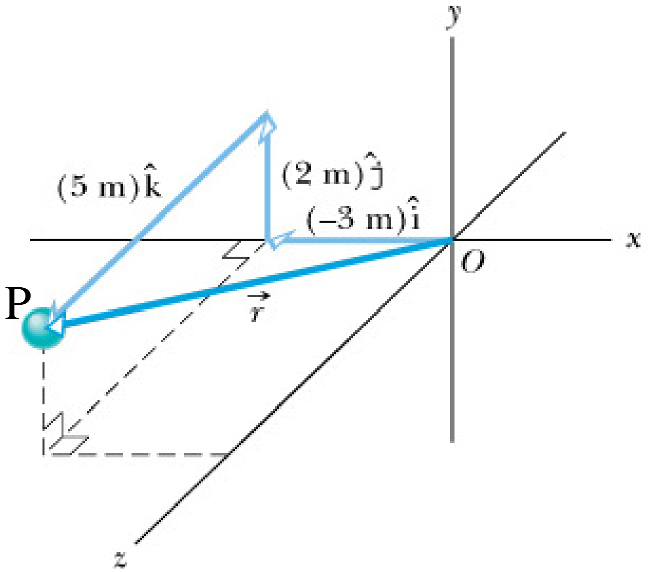




The Position Vector in 3D

The position vector $\vec{r}$ of a particle is defined as a vector whose tail is at a reference point (usually the origin *O*) and its tip is at the particle at point *P*.

Example: The position vector in the figure is $\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}$



$$\vec{r} = \left(-3\hat{\mathrm{i}} + 2\hat{\mathrm{j}} + 5\hat{\mathrm{k}}\right)m$$

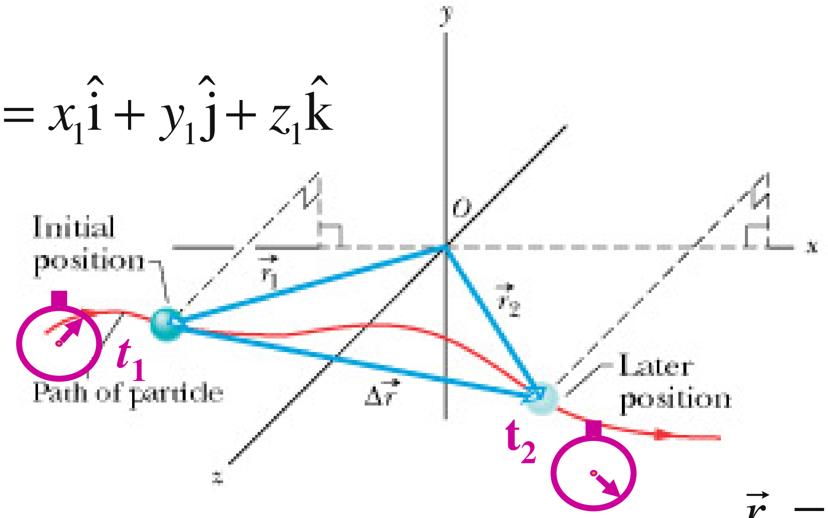
Displacement Vector

For a particle that changes position vector from $\vec{r}_1$ to $\vec{r}_2$ we define the displacement vector $\Delta\vec{r}$ as follows: $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$.

The displacement $\Delta\vec{\mathrm{r}}$ can then be written as

$$\Delta\vec{r} = (x_2 - x_1)\hat{\mathrm{i}} + (y_2 - y_1)\hat{\mathrm{j}} + (z_2 - z_1)\hat{\mathrm{k}} = \Delta x\hat{\mathrm{i}} + \Delta y\hat{\mathrm{j}} + \Delta z\hat{\mathrm{k}}$$

$$\vec{r}_1 = x_1\hat{\mathrm{i}} + y_1\hat{\mathrm{j}} + z_1\hat{\mathrm{k}}$$



$$\Delta x = x_2 - x_1$$

$$\Delta y = y_2 - y_1$$

$$\Delta z = z_2 - z_1$$

$$\vec{r}_2 = x_2\hat{\mathrm{i}} + y_2\hat{\mathrm{j}} + z_2\hat{\mathrm{k}}$$

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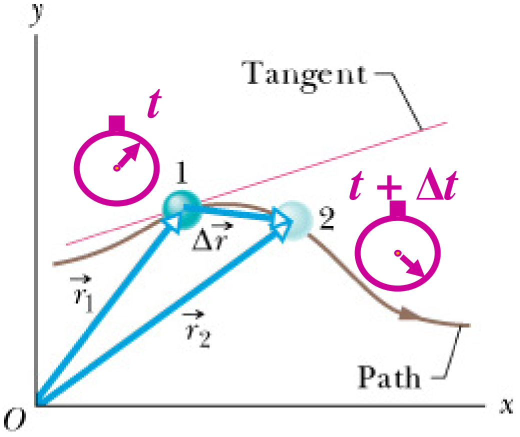
Average and Instantaneous Velocity

Following the same approach as in Chapter 2 we define the average velocity as

$$\text{average velocity} = \frac{\text{displacement}}{\text{time interval}} \quad \Rightarrow \quad \vec{v}_{\text{avg}} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x\hat{\mathrm{i}} + \Delta y\hat{\mathrm{j}} + \Delta z\hat{\mathrm{k}}}{\Delta t} = \frac{\Delta x\hat{\mathrm{i}}}{\Delta t} + \frac{\Delta y\hat{\mathrm{j}}}{\Delta t} + \frac{\Delta z\hat{\mathrm{k}}}{\Delta t}$$

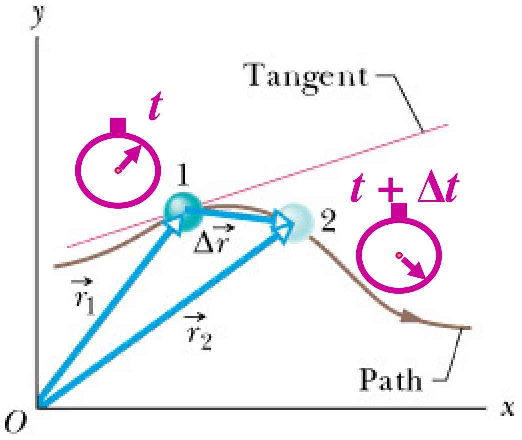


We define the instantaneous velocity (or more simply the velocity) as the limit:

$$\boxed{\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}}$$

Velcocity Components

The three velocity components are given by the equations



$$\boxed{\vec{v} = \frac{d\vec{r}}{dt}} \quad \left\{ \begin{array}{l} \boxed{v_x = \dfrac{dx}{dt}} \\ \boxed{v_y = \dfrac{dy}{dt}} \\ \boxed{v_z = \dfrac{dz}{dt}} \end{array} \right.$$

Average and Instantaneous Acceleration

The average acceleration is defined as:

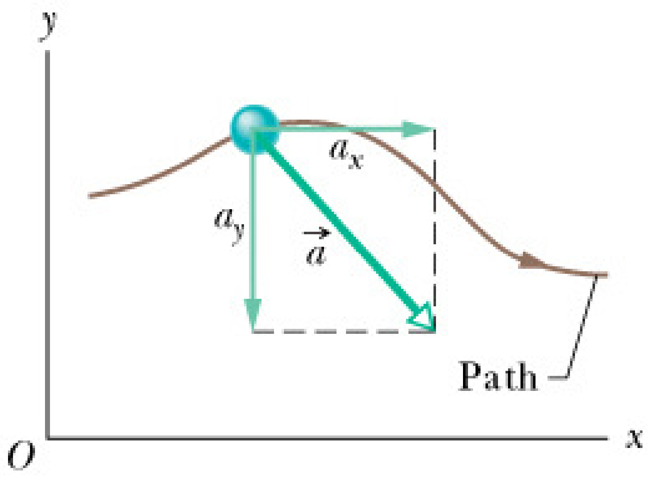
$$\vec{a}_{\text{avg}} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}$$

We define the instantaneous acceleration as the limit:

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt}}$$

The three acceleration components are given by the equations:

$$\boxed{a_x = \frac{dv_x}{dt}} \quad \boxed{a_y = \frac{dv_y}{dt}} \quad \boxed{a_z = \frac{dv_z}{dt}}$$



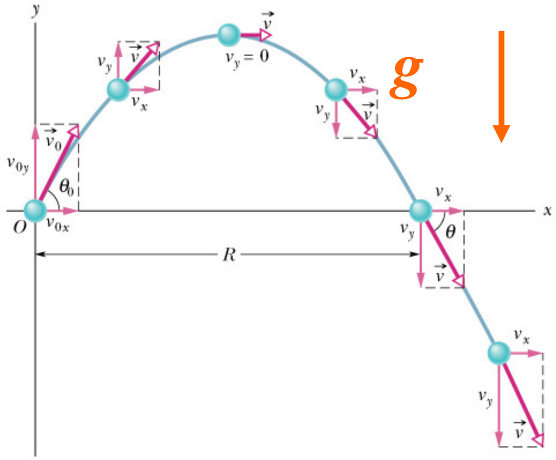
Note: Unlike velocity, the acceleration vector does not have any specific relationship with the path.

Projectile Motion

The motion of an object in a vertical plane under the influence of gravitational force is known as "projectile motion."

The projectile is launched with an initial velocity $\vec{v}_0$. Therefore the horizontal and vertical velocity components are given by:

$$v_{0x} = v_0 \cos\theta_0 \qquad v_{0y} = v_0 \sin\theta_0$$



Projectile motion will be analyzed in a horizontal and a vertical motion along the *x*- and *y*-axes, respectively. These two motions are independent of each other. Motion along the *x*-axis has zero acceleration. Motion along the *y*-axis has uniform acceleration $a_y = -g$.

Basic Projectile Motion Equations

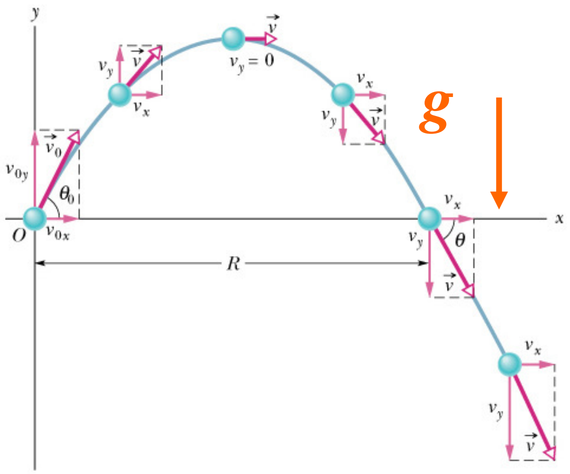
Horizontal Motion: $a_x = 0$ The velocity along the *x*-axis does not change:

$v_x = v_0 \cos\theta_0$ (eq. 1) $x = x_o + (v_0 \cos\theta_0)t$ (eq. 2)

Vertical Motion: $a_y = -g$ Along the *y*-axis the projectile is in free fall

$v_y = v_0 \sin\theta_0 - gt$ (eq. 3) $y = y_o + (v_0 \sin\theta_0)t - \dfrac{gt^2}{2}$ (eq. 4)

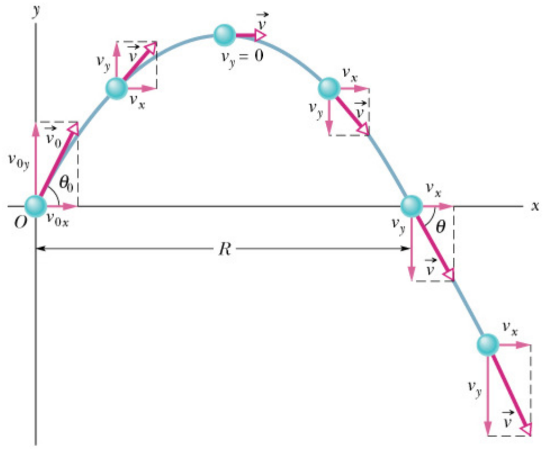
If we eliminate t between equations 3 and 4 we get $v_y^2 - (v_0 \sin\theta_0)^2 = -2g(y - y_o)$.



Here x_o and y_o are the coordinates of the launching point. For many problems the launching point is taken at the origin. In this case $x_o = 0$ and $y_o = 0$.

Note: In this analysis of projectile motion we neglect the effects of air resistance.

The Parabolic Trajectory



The equation of the path:

$$x = (v_0 \cos\theta_0)t \qquad \text{(eq. 2)} \qquad\qquad y = (v_0 \sin\theta_0)t - \frac{gt^2}{2} \qquad \text{(eq. 4)}$$

If we eliminate t between equations 2 and 4 we get:

$$y = (\tan\theta_0)x - \frac{g}{2(v_0\cos\theta_0)^2}x^2.$$ This equation describes the path of the motion.

The path equations has the form: $y = ax + bx^2$. This is the equation of a parabola.

The Range of a Projectile - R

Horizontal Range: The distance *OA* is defined as the horizontal range *R*

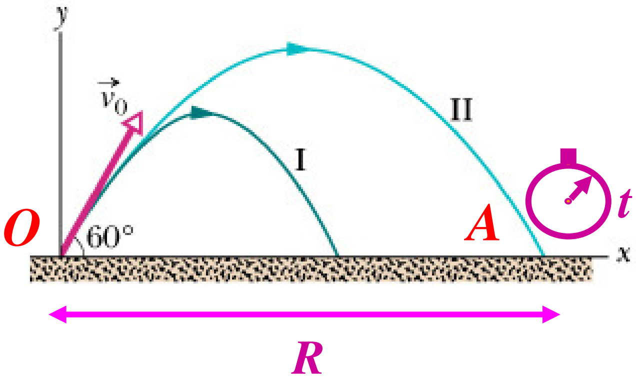
At point *A* we have: $y = 0$. From equation 4 we have:

$(v_0 \sin\theta_0)t - \frac{gt^2}{2} = 0 \rightarrow t\left(v_0 \sin\theta_0 - \frac{gt}{2}\right) = 0$. This equation has two solutions:

Solution 1. $t = 0$. This solution corresponds to point *O* and is of no interest.

Solution 2. $v_0 \sin\theta_0 - \frac{gt}{2} = 0$. This solution corresponds to point *A*.

From solution 2 we get $t = \frac{2v_0 \sin\theta_0}{g}$. If we substitute t in eq. 2 we get



$$R = \frac{v_o^2 \sin 2\theta}{g}$$

Maximum Height of a Projectile - H

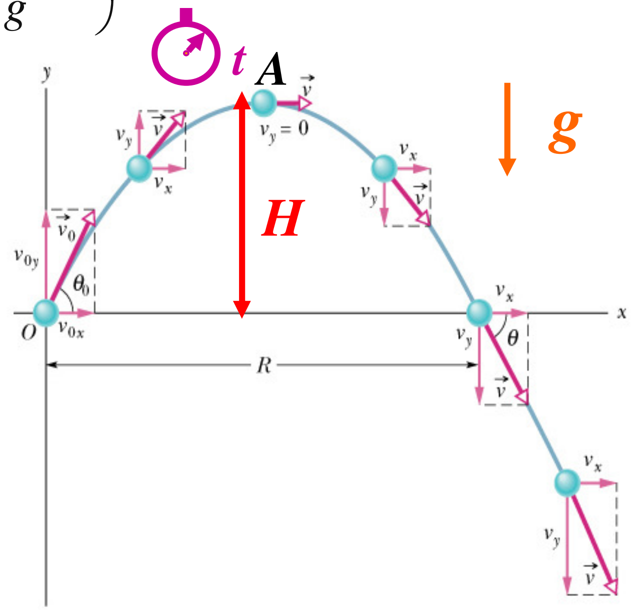
The y-component of the projectile velocity is $v_y = v_0 \sin\theta_0 - gt$.

At point A: $v_y = 0 \quad \rightarrow \; v_0 \sin\theta_0 - gt \rightarrow t = \dfrac{v_0 \sin\theta_0}{g}$

$$H = y(t) = (v_0 \sin\theta_0)\,t - \frac{gt^2}{2} = (v_0 \sin\theta_0)\frac{v_0 \sin\theta_0}{g} - \frac{g}{2}\left(\frac{v_0 \sin\theta_0}{g}\right)^2 \rightarrow$$

$$H = \frac{v_0^2 \sin^2\theta_0}{2g}$$

$$H = \frac{v_o^2 \sin^2\theta}{2g}$$



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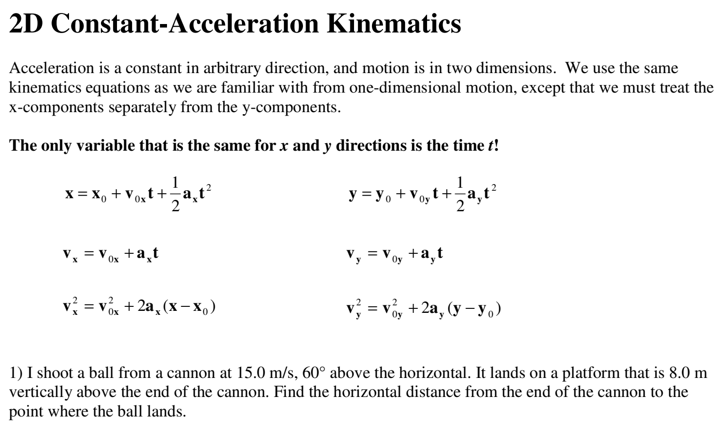
Activity – 2D Kinematics Problems

- **Cannonball Motion**
  - I shoot a ball from a cannon at 15.0 m/s, 60° above the horizontal. It lands on a platform that is 8.0 m vertically above the end of the cannon. Find the horizontal distance from the end of the cannon to the point. where the ball lands.



2D Constant-Acceleration Kinematics

Acceleration is a constant in arbitrary direction, and motion is in two dimensions. We use the same kinematics equations as we are familiar with from one-dimensional motion, except that we must treat the x-components separately from the y-components.

The only variable that is the same for *x* and *y* directions is the time *t*!

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_{0x}\mathbf{t} + \frac{1}{2}\mathbf{a}_x\mathbf{t}^2 \qquad \mathbf{y} = \mathbf{y}_0 + \mathbf{v}_{0y}\mathbf{t} + \frac{1}{2}\mathbf{a}_y\mathbf{t}^2$$

$$\mathbf{v}_x = \mathbf{v}_{0x} + \mathbf{a}_x\mathbf{t} \qquad \mathbf{v}_y = \mathbf{v}_{0y} + \mathbf{a}_y\mathbf{t}$$

$$\mathbf{v}_x^2 = \mathbf{v}_{0x}^2 + 2\mathbf{a}_x(\mathbf{x} - \mathbf{x}_0) \qquad \mathbf{v}_y^2 = \mathbf{v}_{0y}^2 + 2\mathbf{a}_y(\mathbf{y} - \mathbf{y}_0)$$

1) I shoot a ball from a cannon at 15.0 m/s, 60° above the horizontal. It lands on a platform that is 8.0 m vertically above the end of the cannon. Find the horizontal distance from the end of the cannon to the point where the ball lands.

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