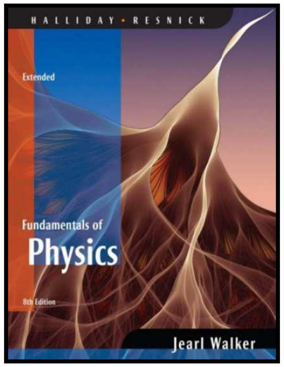
Workshop Physics 1017 - 311

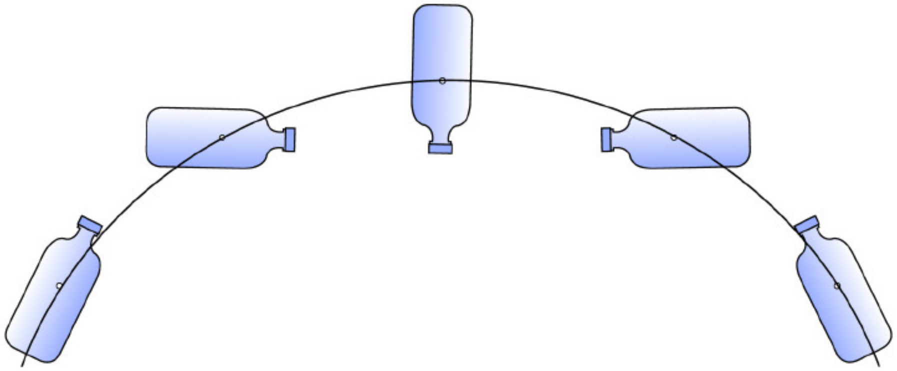
University Physics I



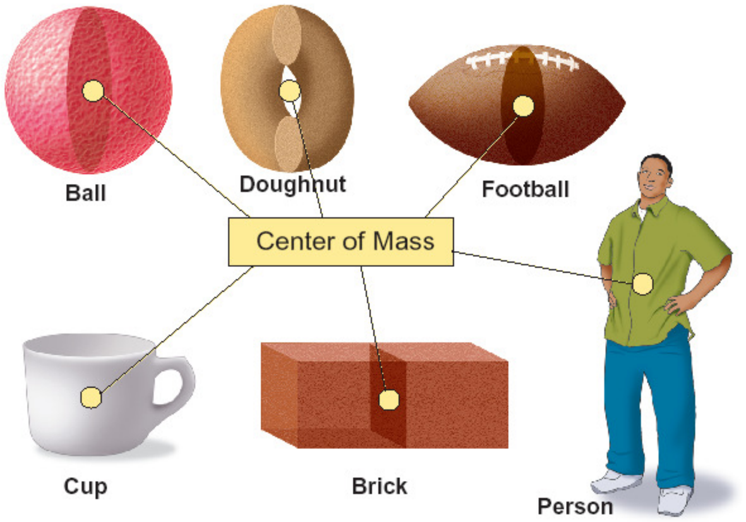
Week 8 : Day 2

Definition of the Center of Mass

- ❑ There are three different axes about which an object will naturally spin.
- ❑ The point at which the three axes intersect is called the center of mass.



Center of Mass



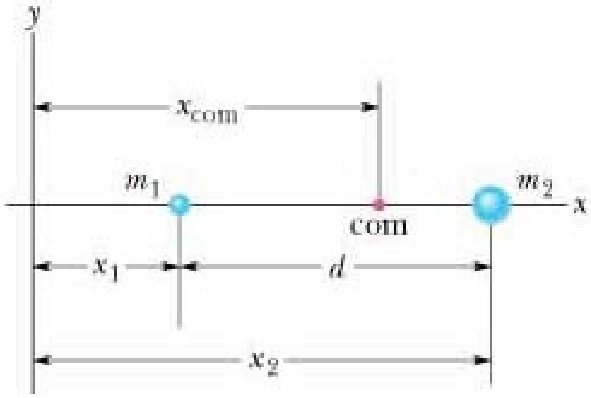
Center of Mass

Goals:

1. Define the Center of mass (com) for a system of particles
2. Calculate the velocity and acceleration of the center of mass
3. Derive the equation of motion for the center of mass.

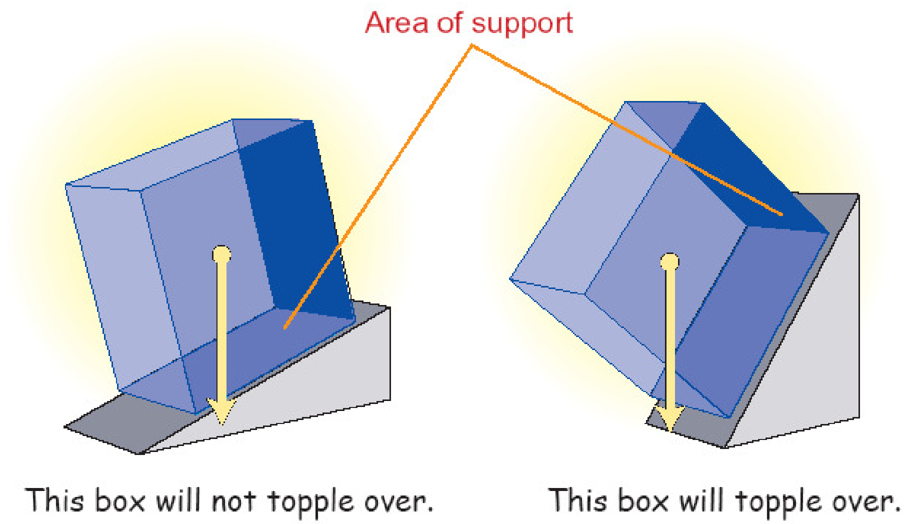
Consider a system of two particles of masses m_1 and m_2 at positions x_1 and x_2, respectively. We define the position of the center of mass (com) as follows:

$$x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$



Balance and Center of Mass

- ☐ For an object to remain upright, its center of gravity must be <u>above</u> its area of support.
- ☐ The area of support includes the entire region surrounded by the actual supports.
- ☐ An object will topple over if its' center of mass is not above its area of support.



Systems of Particles

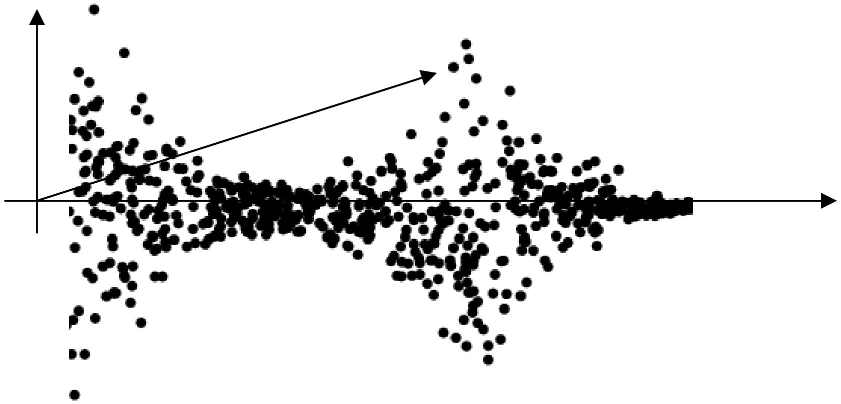
We can generalize the above definition for a system of n particles as follows:

$$x_{\text{com}} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots + m_nx_n}{m_1 + m_2 + m_3 + \ldots + m_n} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots + m_nx_n}{M} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$$

Here M is the total mass of all the particles $M = m_1 + m_2 + m_3 + \ldots + m_n$.

We can further generalize the definition for the center of mass of a system of particles in three-dimensional space. We assume that the ith particle (mass m_i) has position vector $\vec{r}_i$.

$$\vec{r}_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$$



Activity - Center of Mass of a Group of Particles

- ❑ **Work in groups**
  - ➢ COM of Earth-Moon
  - ➢ Three particle system

Center of Mass of a Group of Particles

At first sight, calculating the motion of an extended object (such as a javelin) or a group of particles (such as debris from an explosion) seems complicated. But we can make life easier by following only the motion of the **center of mass** of the system. Thus, we can determine the motion of a javelin by applying Newton's Laws to its center of mass, rather than calculating the motion of each individual point.

In 2-dimensions, the center of mass of a group of N point masses (particles) is

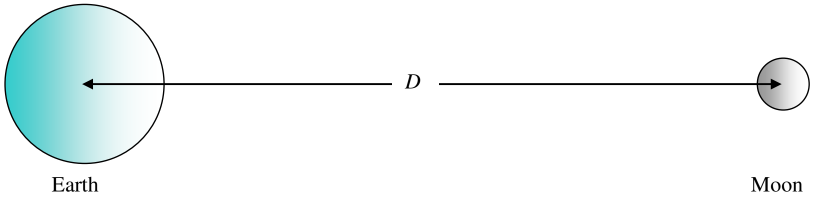
$$\vec{R}_{CoM} = X_{CoM}\hat{\imath} + Y_{CoM}\hat{\jmath}$$

where

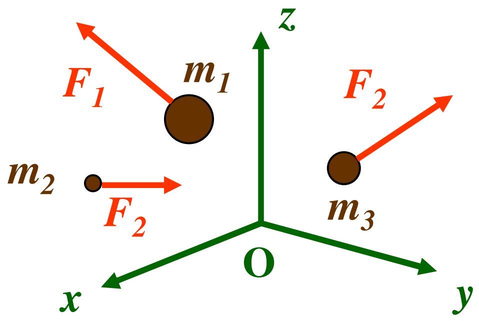
$$X_{CoM} = \sum_{i=1}^{N} x_i m_i \Big/ \sum_{i=1}^{N} m_i \text{ and } Y_{CoM} = \sum_{i=1}^{N} y_i m_i \Big/ \sum_{i=1}^{N} m_i$$

are the x- and y- components of the center of mass, respectively.
Even if you are not actually dealing with particles, it is often possible to treat objects as if they are point masses, if they are uniform and symmetric (e.g., its obvious that the center of mass of a uniform disk must be at its center), or if you are trying to find the CoM of objects which are small compared to their distance apart (e.g., Earth and Moon).

OK, so here's the Earth and Moon. Where is the CoM?
You will find the relevant data in the Appendix in your textbook. Take the origin of the x-axis to be the center of the Earth. Is the CoM inside or outside the Earth?



Newton’s Second Law for a System of Particles



Consider a system of n particles of masses $m_1, m_2, m_{3,} \ldots, m_n$ and position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n$, respectively.

The position vector of the center of mass is given by

$M\vec{r}_{\text{com}} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots + m_n\vec{r}_n$. We take the time derivative of both sides →

$$M\frac{d}{dt}\vec{r}_{\text{com}} = m_1\frac{d}{dt}\vec{r}_1 + m_2\frac{d}{dt}\vec{r}_2 + m_3\frac{d}{dt}\vec{r}_3 + \ldots + m_n\frac{d}{dt}\vec{r}_n \rightarrow$$

$M\vec{v}_{\text{com}} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots + m_n\vec{v}_n$. Here $\vec{v}_{\text{com}}$ is the velocity of the com and $\vec{v}_i$ is the velocity of the ith particle. We take the time derivative once more →

$$M\frac{d}{dt}\vec{v}_{\text{com}} = m_1\frac{d}{dt}\vec{v}_1 + m_2\frac{d}{dt}\vec{v}_2 + m_3\frac{d}{dt}\vec{v}_3 + \ldots + m_n\frac{d}{dt}\vec{v}_n \rightarrow$$

$M\vec{a}_{\text{com}} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots + m_n\vec{a}_n$. Here $\vec{a}_{\text{com}}$ is the acceleration of the com and $\vec{a}_i$ is the acceleration of the ith particle.

Center of Mass Movement

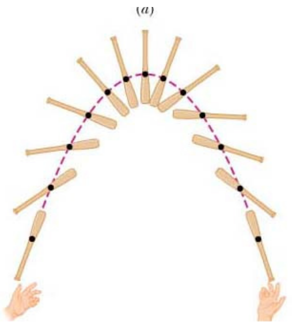
The position vector for the center of mass is given by the equation $\vec{r}_{\text{com}} = \dfrac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$.

The position vector can be written as $\vec{r}_{\text{com}} = x_{\text{com}}\hat{\text{i}} + y_{\text{com}}\hat{\text{j}} + z_{\text{com}}\hat{\text{k}}$.

The components of $\vec{r}_{\text{com}}$ are given by the equations

$$x_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \qquad y_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i \qquad z_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

The center of mass of a system of particles moves as though all the system's mass were concentrated there, and that the vector sum of all the external forces were applied there



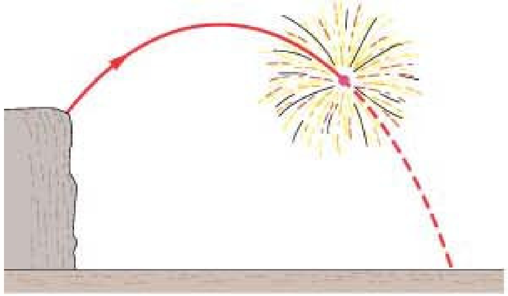
The equation of motion for the center of mass becomes $M\vec{a}_{\text{com}} = \vec{F}_{\text{net}}$.

In terms of components we have:

$$F_{\text{net},x} = Ma_{\text{com},x} \qquad F_{\text{net},y} = Ma_{\text{com},y} \qquad F_{\text{net},z} = Ma_{\text{com},z}$$

Center of Mass Example: Exploding Object

$$M\vec{a}_{\mathrm{com}} = \vec{F}_{\mathrm{net}}$$



The equations above show that the center of mass of a system of particles moves as though all the system's mass were concentrated there, and that the vector sum of all the external forces were applied there. A dramatic example is given in the figure. In a fireworks display a rocket is launched and moves under the influence of gravity on a parabolic path (projectile motion). At a certain point the rocket explodes into fragments. If the explosion had not occurred, the rocket would have continued to move on the parabolic trajectory (dashed line). The forces of the explosion, even though large, are all internal and as such cancel out. The only external force is that of gravity and this remains the same before and after the explosion. This means that the center of mass of the fragments follows the same parabolic trajectory that the rocket would have followed had it not exploded.

$$F_{\mathrm{net},x} = Ma_{\mathrm{com},x}$$
$$F_{\mathrm{net},y} = Ma_{\mathrm{com},y}$$
$$F_{\mathrm{net},z} = Ma_{\mathrm{com},z}$$

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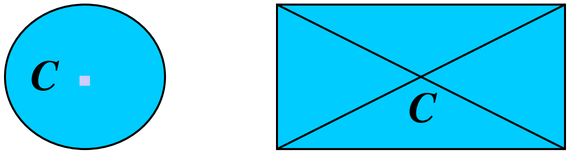
Center of Mass for Continuous Bodies

The sums that are used for the calculation of the center of mass of systems with discrete distribution of mass become integrals:

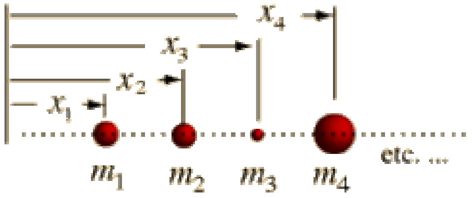
$$x_{\text{com}} = \frac{1}{M}\int x\,dm \qquad y_{\text{com}} = \frac{1}{M}\int y\,dm \qquad z_{\text{com}} = \frac{1}{M}\int z\,dm$$

- ☐ **Here the mass element dm may be expressed in a variety of ways depending on the geometry**
  - 1-D: $dm=\lambda dl$
  - 2-D: $dm=\sigma\, dA$
  - 3-D: $dm = \rho\, dV$
- ☐ **If the mass is uniformly distributed the center of mass of many objects can be found by inspection**
  - A table of "centroids" is often used in engineering applications



Center of Mass Example: Continuous Bodies

For a continuous distribution of mass, the expression for the center of mass of a collection of particles :



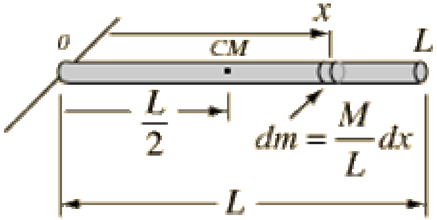
$$x_{cm} = \frac{\sum_{i=1}^{N} m_i x_i}{M}$$

becomes an infinite sum and is expressed in the form of an integral

$$x_{cm} = \lim_{\Delta m \to 0} \frac{\sum_{i=1}^{N} \Delta m_i x_i}{M} = \frac{\int_0^M x\,dm}{M}$$

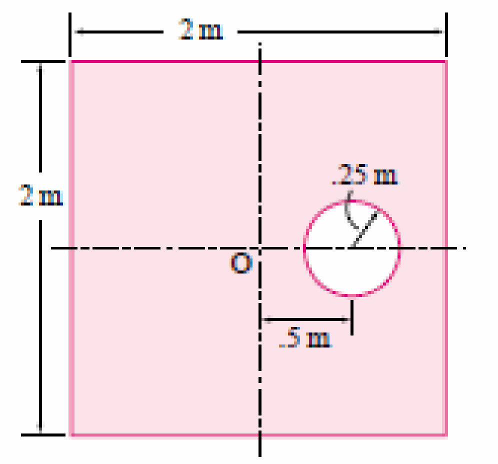
For the case of a uniform rod this becomes

$$x_{cm} = \frac{\int_0^L x \frac{M}{L} dx}{M} = \frac{1}{L} \frac{x^2}{2} \Big|_{x=0}^{x=L} = \frac{L}{2}$$



Center of Mass Example: Regular Objects

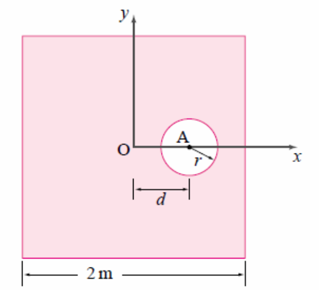
- ❑ **Consider a typical machine part**
  - ➢ A circular hole is cut in a square plate
  - ➢ Therefore the COM is shifted



$$\begin{aligned} m_1 x_{cm} + m_2 x_A &= m x_O = 0 \\ \Rightarrow \quad x_{cm} &= -\frac{m_2}{m_1} x_A . \end{aligned}$$

  - ➢ Since the plates are uniform the masses are similar to the areas:

$$\frac{m_2}{m_1} = \frac{\pi r^2}{\ell^2 - \pi r^2} = \frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi} .$$



$$\begin{aligned} x_{cm} &= -\frac{m_2}{m_1} d = -\frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi} d \\ &= -\frac{\pi}{\left(\frac{2\,\mathrm{m}}{.25\,\mathrm{m}}\right)^2 - \pi} \cdot 0.5\,\mathrm{m} \\ &= -25.81 \times 10^{-3}\,\mathrm{m} = -25.81\,\mathrm{mm} \end{aligned}$$

Center of Mass Example: Direct Integration

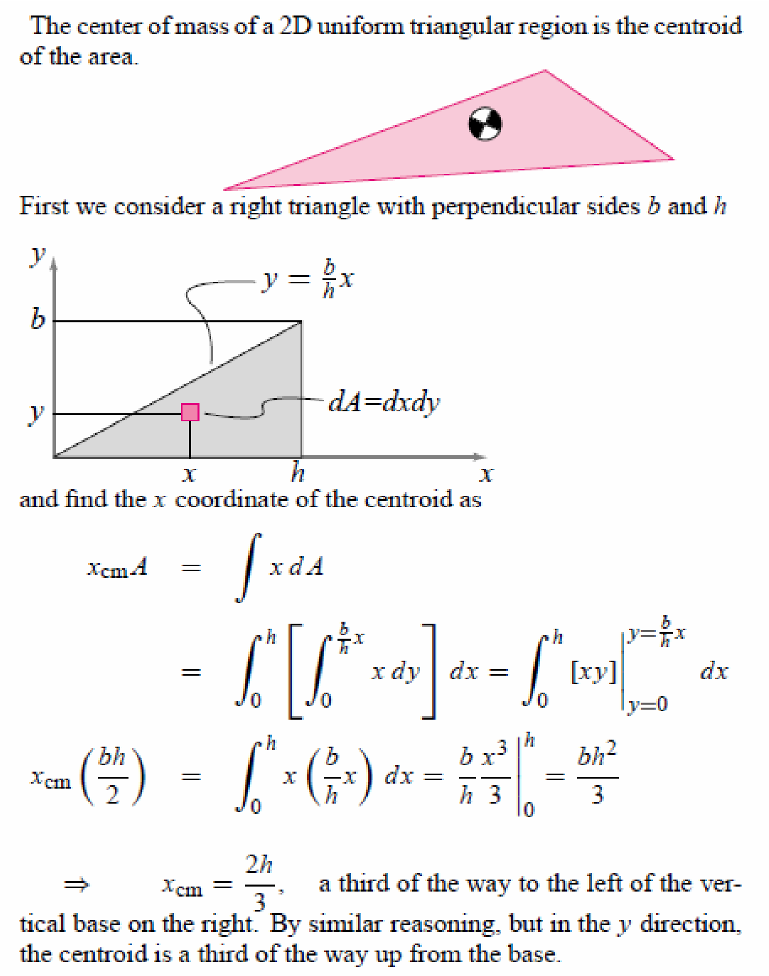
- ❑ **Begin with the definition of the center of mass**

$$\vec{r}_{COM} = \int \vec{r}\, dm$$

- ➢ Integrate over the entire area

$$x_{COM} = \int x \underbrace{\frac{M}{A}\, dxdy}_{dm} = \frac{2}{3}h$$

$$y_{COM} = \int y \underbrace{\frac{M}{A}\, dxdy}_{dm} = ?$$



The center of mass of a 2D uniform triangular region is the centroid of the area.

First we consider a right triangle with perpendicular sides b and h

and find the x coordinate of the centroid as

$$x_{cm}A = \int x\, dA$$

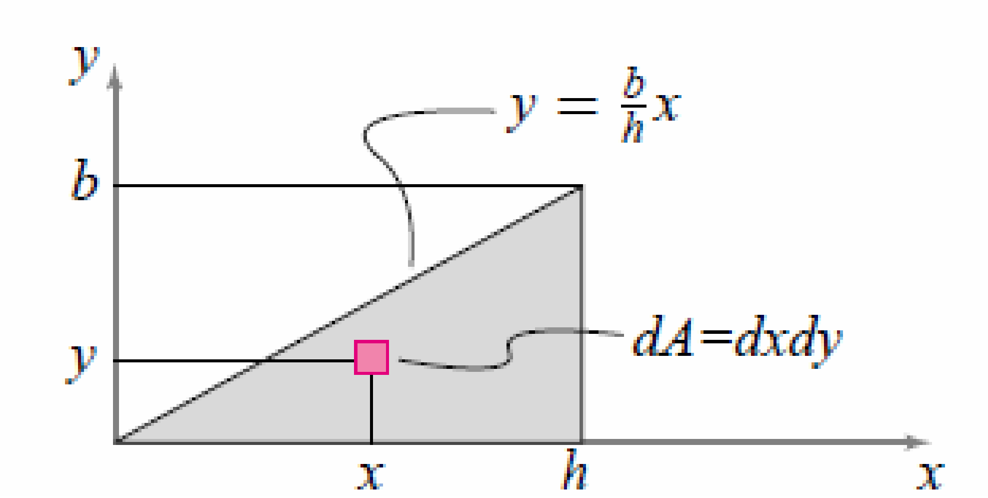
$$= \int_0^h \left[\int_0^{\frac{b}{h}x} x\, dy\right] dx = \int_0^h [xy]\Big|_{y=0}^{y=\frac{b}{h}x} dx$$

$$x_{cm}\left(\frac{bh}{2}\right) = \int_0^h x\left(\frac{b}{h}x\right) dx = \frac{b}{h}\frac{x^3}{3}\Big|_0^h = \frac{bh^2}{3}$$

$\Rightarrow \quad x_{cm} = \frac{2h}{3}$, a third of the way to the left of the vertical base on the right. By similar reasoning, but in the y direction, the centroid is a third of the way up from the base.

Activity – Center of Mass of Triangle

- ❑ **Set up the integral**
  - ➢ Determine y_{com}



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