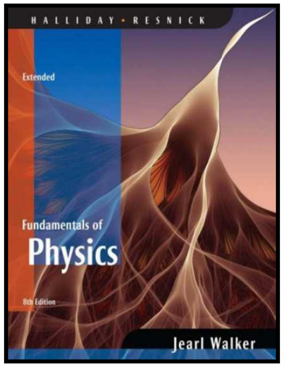
University Physics II

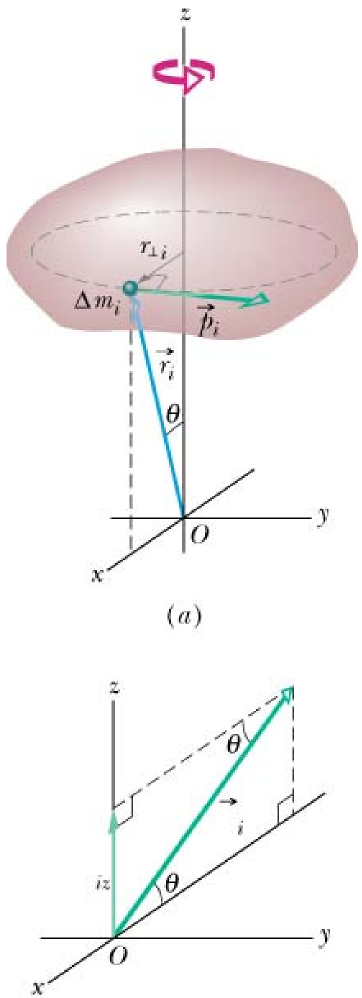


Week 6 : Day 1

Outline

- **Angular Momentum**
  - Relation to torque
  - Conservation
- **Equilibrium**
  - Statics - Problem recipe
- **Elasticity**
  - Stress and strain
- **Simple Harmonic Oscillator**
  - The SHO system
  - Examples…

Angular Momentum of a Rigid Body



We take the z-axis to be the fixed rotation axis. We will determine the z-component of the net angular momentum. The body is divided into n elements of mass Δm_i that have a position vector $\vec{r}_i$.

The angular momentum $\vec{\ell}_i$ of the ith element is $\vec{\ell}_i = \vec{r}_i \times \vec{p}_i$.

Its magnitude is $\ell_i = r_i p_i (\sin 90°) = r_i \Delta m_i v_i$. The z-compoment ℓ_{iz} of ℓ_i is $\ell_{iz} = \ell_i \sin\theta = (r_i \sin\theta)(\Delta m_i v_i) = r_{\perp i} \Delta m_i v_i$.

The z-component of the angular momentum L_z is the sum:

$$L_z = \sum_{i=1}^{n} \ell_{iz} = \sum_{i=1}^{n} r_{\perp i} \Delta m_i v_i = \sum_{i=1}^{n} r_{\perp i} \Delta m_i (\omega r_{i\perp}) = \omega \left(\sum_{i=1}^{n} \Delta m_i r_{i\perp}^2 \right)$$

The sum $\sum_{i=1}^{n} \Delta m_i r_{i\perp}^2$ is the rotational inertia I of the rigid body.

Thus: $L_z = I\omega$.

$$L_z = I\omega$$

$$\vec{\tau} = \vec{\tau}$$

$$\Rightarrow \vec{r} \times \vec{F} = I\vec{\alpha}$$

$$\Rightarrow \vec{r} \times \frac{d}{dt}\vec{p} = I\frac{d}{dt}\vec{\omega}$$

$$\Rightarrow \vec{r} \times \vec{p} = I\vec{\omega}$$

$$\Rightarrow \vec{L} = I\vec{\omega}$$

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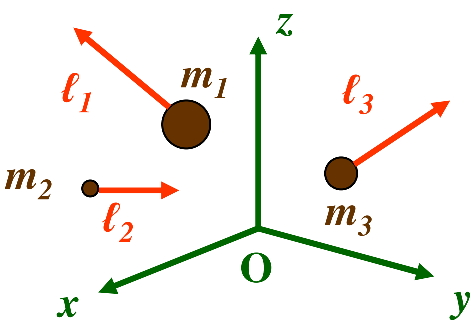
Relating $\vec{\tau}$ and $\vec{L}$



The Angular Momentum of a System of Particles

We will now explore Newton's second law in angular form for a system of n particles that have angular momentum $\vec{\ell}_1, \vec{\ell}_2, \vec{\ell}_3, \ldots, \vec{\ell}_n$.

The angular momentum $\vec{L}$ of the system is $\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \ldots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i$.

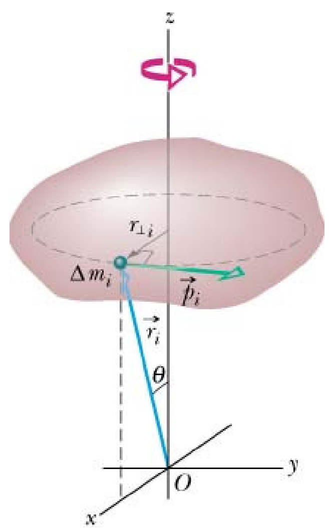
The time derivative of the angular momentum is $\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{\ell}_i}{dt}$.

$$\vec{\tau} = I\vec{\alpha}$$

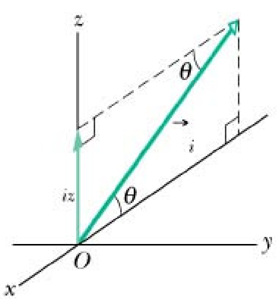
$$\Rightarrow \vec{\tau} = I\frac{d}{dt}\vec{\omega} = \frac{d}{dt}\left(I\vec{\omega}\right)$$

$$\Rightarrow \vec{\tau} = \frac{d}{dt}\vec{L}$$

Conservation of $\vec{L}$



(a)



(b)

For any system of particles (including a rigid body) Newton's second law in angular form is $\dfrac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}}$.

If the net external torque $\vec{\tau}_{\text{net}} = 0$ then we have: $\dfrac{d\vec{L}}{dt} = 0 \rightarrow$ $\vec{L} =$ a constant. This result is known as the law of the conservation of angular momentum. In words:

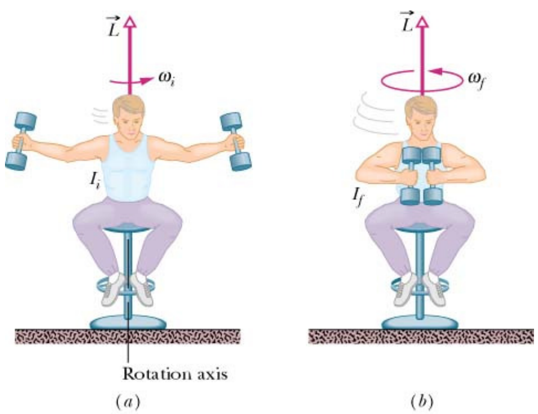
$$\begin{pmatrix}\text{Net angular momentum} \\ \text{at some initial time } t_i\end{pmatrix} = \begin{pmatrix}\text{Net angular momentum} \\ \text{at some later time } t_f\end{pmatrix}$$

In equation form: $\vec{L}_i = \vec{L}_f$

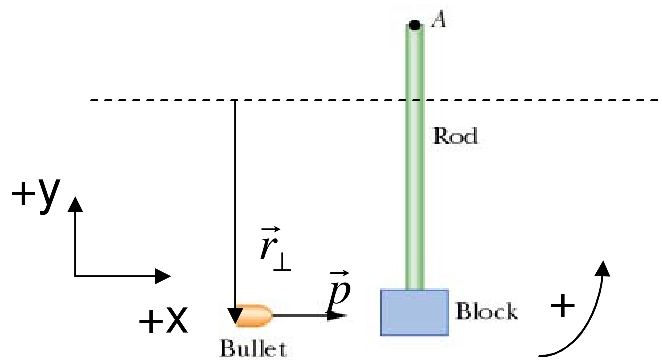
Note: If the component of the external torque along a certain axis is equal to zero, then the component of the angular momentum of the system along this axis cannot change.

Angular Momentum Problems

- **Angular momentum**
  - Has two terms… $\Rightarrow \vec{L} = \vec{r} \times \vec{p} + I\vec{\omega}$, $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$
  - Just like energy…



$$\vec{L}_i = \vec{L}_f$$
$$I_i\omega_i\hat{k} = I_f\omega_f\hat{k}$$
$$\Rightarrow \omega_f = \frac{I_i}{I_f}\omega_i$$



$$\vec{L}_i = \vec{L}_f$$
$$\vec{r} \times \vec{p} = I\vec{\omega}$$
$$\left(-L\hat{j}\right) \times m_{bullet}v_{bullet}\hat{i} = \left[\left(m_{bullet} + m_{Block}\right)L^2 + \tfrac{1}{3}m_{rod}\left(L - \tfrac{h}{2}\right)^2\right]\left(+\omega\hat{k}\right)$$
$$\Rightarrow v_{bullet} = +\frac{\left[m_{bullet} + m_{Block} + \frac{1}{3}\left(1 - \frac{h}{2L}\right)^2 m_{rod}\right]}{m_{bullet}}\omega L$$

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Conditions of Equilibrium

In Chapter 9 we calculated the rate of change for the linear momentum of the center of mass of an object, $\frac{d\vec{P}}{dt} = \vec{F}_{\text{net}}$. If an object is in translational equilibrium then

$\vec{P} =$ constant and thus $\frac{d\vec{P}}{dt} = 0 \rightarrow \quad \vec{F}_{\text{net}} = 0.$

In Chapter 11 we analyzed rotational motion and saw that Newton's second law takes the form $\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}}$. For an object in rotational equilibrium we have: $\vec{L} =$ constant

$\rightarrow \frac{d\vec{L}}{dt} = 0 \rightarrow \quad \vec{\tau}_{\text{net}} = 0.$

1. The vector sum of all the external forces on the body must be zero.
2. The vector sum of all the external torques that act on the body measured about any point must be zero.

Problem Recipe

1. Draw a force diagram. (Label the axes.)
2. Choose a **convenient** origin O

   have one of the unknown force acting at *O*

3. Sign of the torque τ for each force:

   - Force induces clockwise (CW) rotation

   + Force induces counterclockwise (CCW) rotation

4. Apply equilibrium conditions:

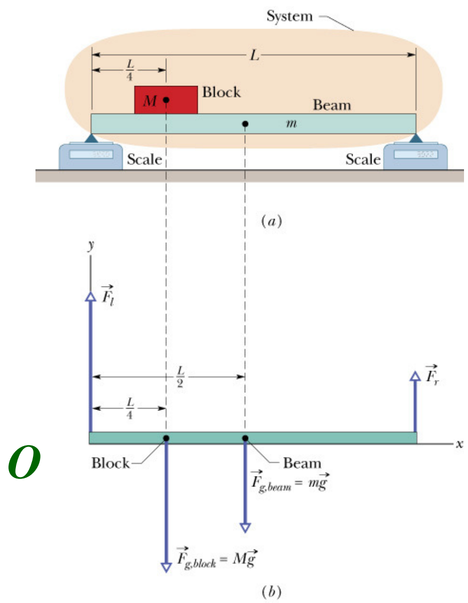
$$F_{\text{net},x} = 0 \qquad F_{\text{net},y} = 0$$

$$\tau_{\text{net},z} = 0$$

5. Make sure that

   number of unknowns = number of equations

Example – Beam Balance



Sample Problem 12-1. A uniform beam of length L and mass m = 1.8 kg is at rest on two scales.
A uniform block of mass M = 2.7 kg is at rest on the beam at a distance $L/4$ from its left end.
Calculate the scales readings:

$$F_{\text{net},y} = F_\ell + F_r - Mg - mg = 0 \quad \text{(eq. 1)}$$

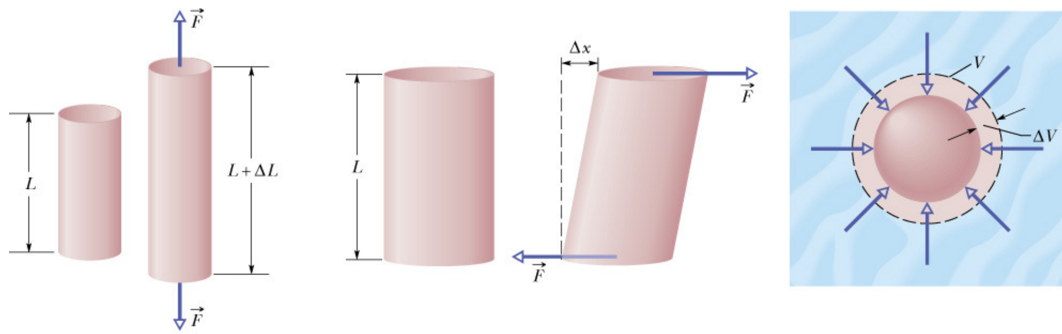
We choose to calculate the torque with respect to an axis through the left end of the beam (point O).

$$\tau_{\text{net},z} = -\left(\frac{L}{4}\right)(mg) - \left(\frac{L}{2}\right)(Mg) + (L)(F_r) = 0 \quad \text{(eq. 2)}$$

From equation 2 we get: $F_r = \dfrac{Mg}{4} + \dfrac{mg}{2} = \dfrac{2.7\times 9.8}{4} + \dfrac{1.8\times 9.8}{2} = 15.44 \approx 15$ N.

We solve equation 1 for $F_\ell \rightarrow F_\ell = Mg + mg - F_r = (2.7+1.8)\times 9.8 - 15.44 = 28.66$ N:
$F_\ell \approx 29$ N.

Stress and Strain

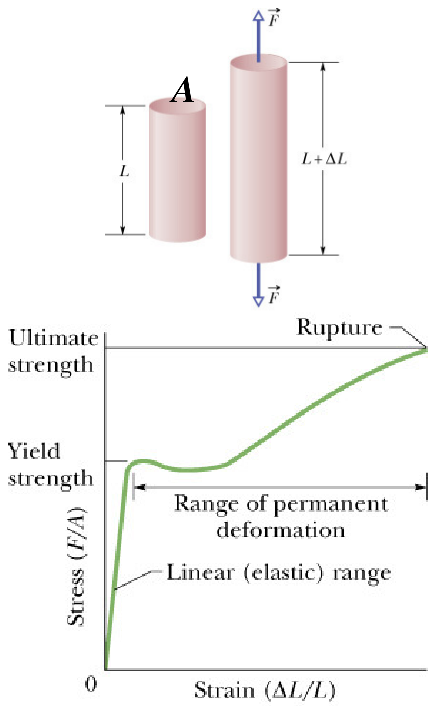


(a) (b) (c)

In the three figures above we show the three ways in which a solid might change its dimensions under the action of external deforming forces. In fig. *a* the cylinder is stretched by forces acting along the cylinder axis. In fig. *b* the cylinder is deformed by forces perpendicular to its axis. In fig. *c* a solid placed in a fluid under high pressure is compressed uniformly on all sides. All three deformation types have stress in common (defined as deforming force per unit area). These stresses are known as tensile/compressive for fig. *a*, shearing for fig. *b*, and hydraulic for fig. *c*. The application of stress on a solid results in strain, which takes different form for the three types of strain. Strain is related to strain via the equation:

$$\text{stress} = \text{modulus} \times \text{strain}$$

Tensile Stress



Tensile stress is defined as the ratio $\frac{F}{A}$ where A is the solid area.

Strain (symbol S) is defined as the ratio $\frac{\Delta L}{L}$ where ΔL is the change in the length L of the cylindrical solid.

Stress is plotted versus strain in the upper figure.

For a wide range of applied stresses the stress-strain relation is linear and the solid returns to its original length when the stress is removed. This is known as the elastic range. If the stress is increased beyond a maximum value known as the yield strength S_y the cylinder becomes permanently deformed. If the stress continues to increase, the cylinder breaks at a stress value known as ultimate strength S_u.

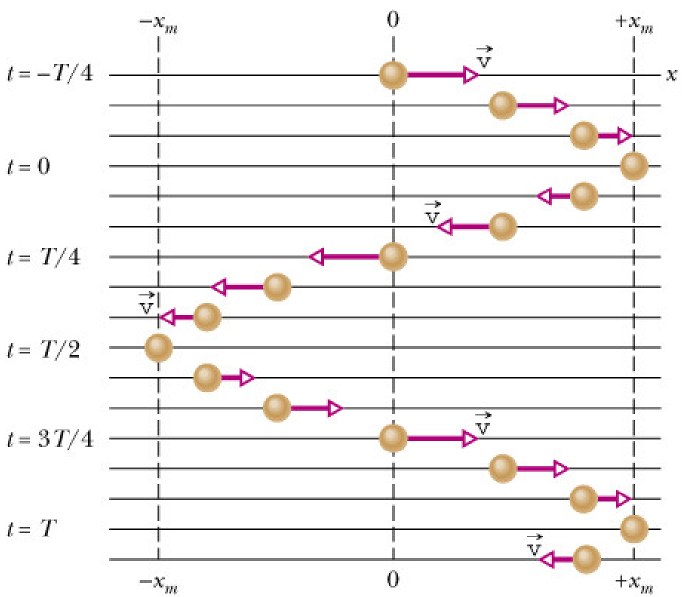
For stresses below S_y (elastic range) stress and strain are connected via the equation

$\frac{F}{A} = E\frac{\Delta L}{L}$. The constant E (modulus) is known as Young's modulus.

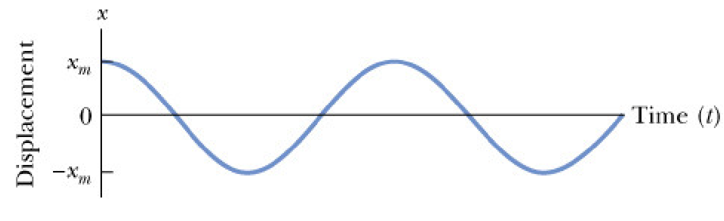
Note: Young's modulus is almost the same for tension and compression.

The ultimate strength S_u may be different.

What is harmonic motion?



(a)



(b)

Simple Harmonic Motion (SHM)

In fig. *a* we show snapshots of a simple oscillating system.

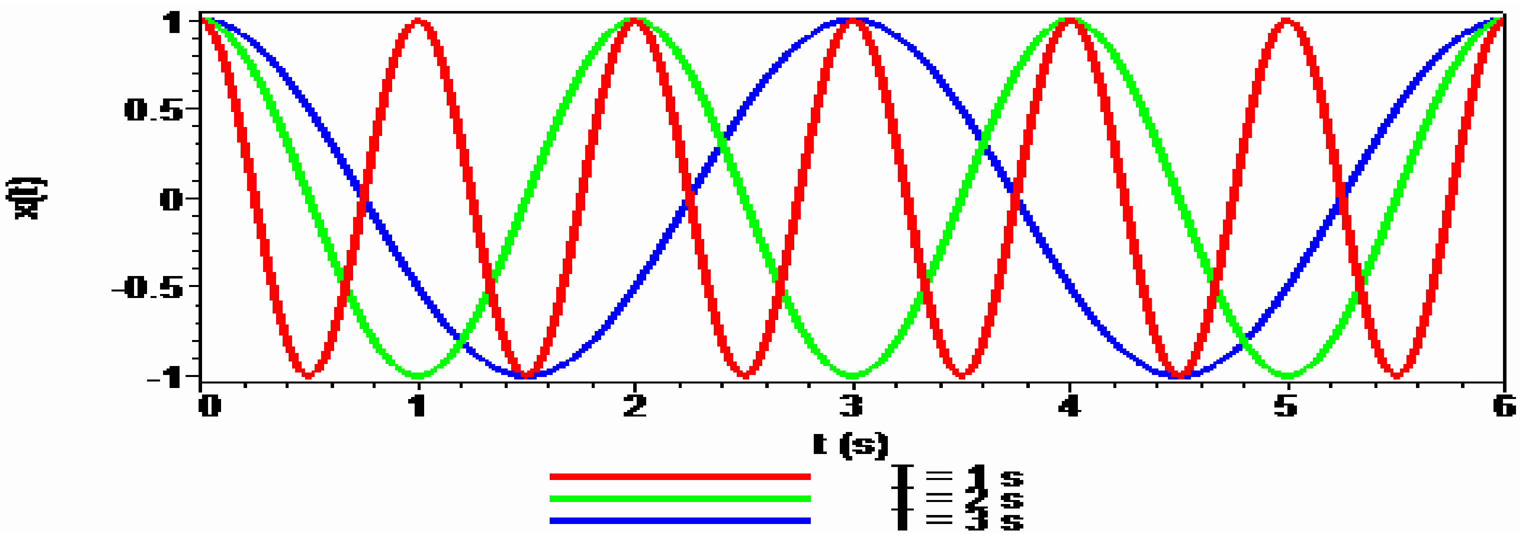
The motion is periodic, i.e., it repeats in time. The time needed to complete one repetition is known as the period (symbol T, units: s). The number of repetitions per unit time is called the frequency (symbol f, unit hertz), $f = \dfrac{1}{T}$.

The displacement of the particle is given by the equation $x(t) = x_m \cos(\omega t + \phi)$.

Parameters of harmonic motion

$x(t) = x_m \cos(\omega t + \phi)$ The quantity x_m is called the amplitude of the motion. It gives the maximum possible displacement of the oscillating object. The quantity ω is called the angular frequency of the oscillator. It is given by the equation

$$\omega = 2\pi f = \frac{2\pi}{T}$$



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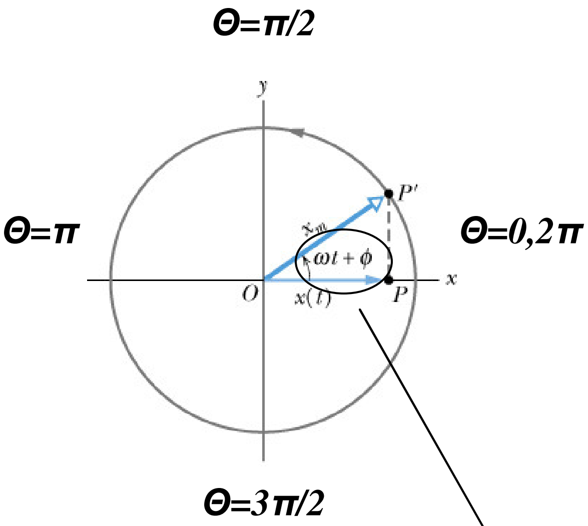
Harmonic and Circular Motion



Consider an object moving on a circular path of radius x_m with a uniform speed v. If we project the position of the moving particle at point P' on the x-axis we get point P. The coordinate of P is $x(t) = x_m \cos(\omega t + \phi)$.

While point P' executes uniform circular motion, its projection P moves along the x-axis with simple harmonic motion.

The phase angle determines the starting position of the harmonic motion…

$$\theta(t) = \omega t + \phi \quad \Rightarrow \quad \theta(t=0) = \phi$$

Equation of Motion

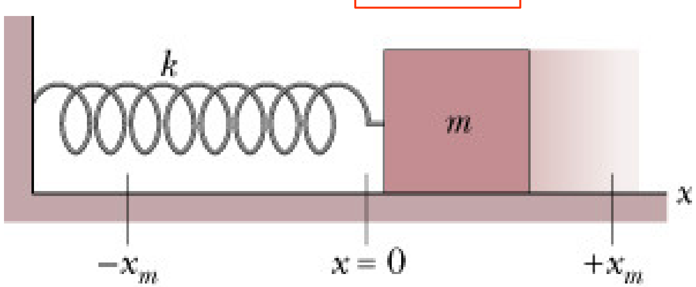
The Force Law for Simple Harmonic Motion

We saw that the acceleration of an object undergoing SHM is $a = -\omega^2 x$.

If we apply Newton's second law we get: $F = ma = -m\omega^2 x = -\left(m\omega^2\right)x.$

Simple harmonic motion occurs when the force acting on an object is proportional to the displacement but opposite in sign. The force can be written as $F = -Cx$ where C is a constant. If we compare the two expressions for F we have

$m\omega^2 = C \rightarrow$ and $T = 2\pi\sqrt{\frac{m}{C}}$

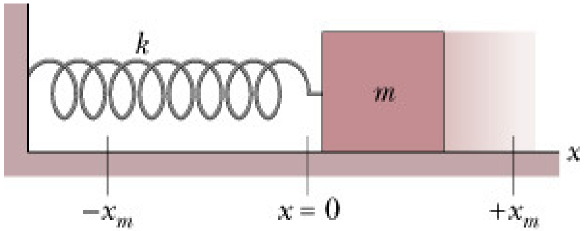


$$m\frac{d^2x}{dt^2} = -Cx$$

The motion is described by a second-order linear differential equation…

Hooke's Law

Robert Hooke, (18 July 1635 – 3 March 1703) was an English natural philosopher who played an important role in the scientific revolution, through both experimental and theoretical work. Hooke is known principally for his law of elasticity (Hooke's Law). He is also remembered for his work as "the father of microscopy" — it was Hooke who coined the term "cell" to describe the basic unit of life.*



Consider the motion of a mass m attached to a spring of spring constant k that moves on a frictionless horizontal floor as shown in the figure.

The net force F on m is given by Hooke's law: $F = -kx$. If we compare this equation with the expression $F = -Cx$ we identify the constant C with the spring constant k. We can then calculate the angular frequency ω and the period T.

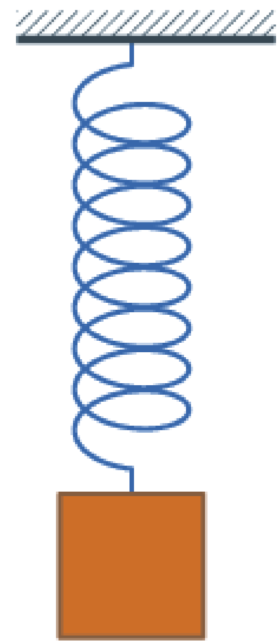
$$\omega = \sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

*http://en.wikipedia.org/wiki/Robert_Hooke ...rmation on Hooke.

The Linear SHO System

- **Begin with Newton's law**
  - $F = ma$
- **Use Hooke's law for the force**
  - $F = -kx$
- **Construct the differential equation**
  - $md^2x/dt^2+kx=0$
- **Substitute the solution $x(t)=x_m cos(\omega t+\varphi)$**
  - $-m\omega^2 x(t)+kx(t)=0$
- **Solve for frequency**
  - $\omega^2 = k/m$
- **Express period of motion**
  - $T = 2\pi\sqrt{m/k}$

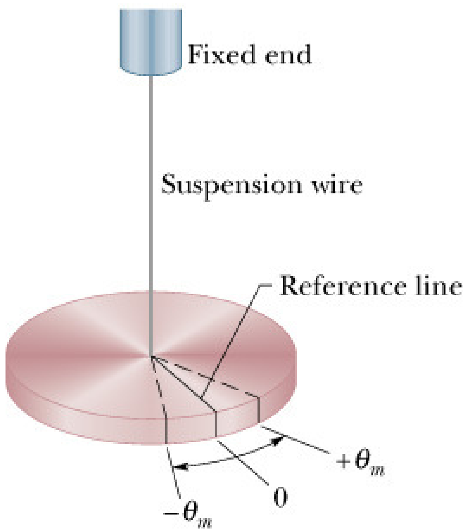
Spring-Mass



The Rotational SHO System

- **Begin with Newton's second law** *(rotational form)*
  - $\tau = I\,\alpha$
- **Use Hooke's law for the force**
  - $\tau = -\kappa\,\theta$
- **Construct the differential equation**
  - $I\,d^2\theta/dt^2 + \kappa\,\theta = 0$
- **Substitute the solution** $\theta(t) = \theta_m \cos(\omega t + \varphi)$
  - $-I\,\omega^2\,\theta(t) + \kappa\,\theta(t) = 0$
- **Solve for frequency**
  - $\omega^2 = \kappa / I$
- **Express period of motion**
  - $T = 2\pi\sqrt{I/\kappa}$

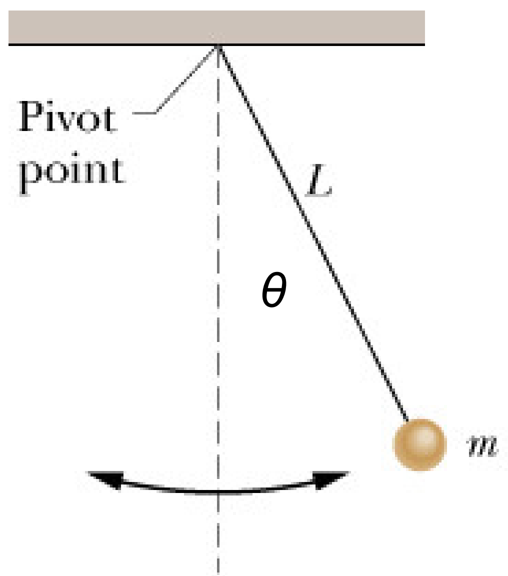
Torsion-Pendulum



The Pendulum SHO System

- **Begin with Newton's second law** *(rotational form)*
  - $\tau = I\,\alpha$
- **Use gravitational force**
  - $\tau = -\,mg\,L\,\sin\theta$
- **Construct the differential equation**
  - $I\,d^2\theta/dt^2 + mgL\,\sin\theta = 0$
- ***Assume small angle approximation***
  - $mL^2\,d^2\theta/dt^2 + mgL\,\theta = 0$
- **Substitute the solution** $\theta(t) = \theta_m \cos(\omega t + \varphi)$
  - $-\,L\,\omega^2\,\theta(t) + g\,\theta(t) = 0$
- **Solve for frequency**
  - $\omega^2 = g/L$
- **Express period of motion**
  - $T = 2\pi\sqrt{L/g}$

Simple Pendulum



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